CONCERNING UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA*

BY

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I. Upper semi-continuous collections of continua which do not separate a plane

A collection of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance† from g less than d then the upper distance of x from g is less than e. The element p of such a collection G is said to be a *limit element* of the subcollection K of G if for every positive number e there exists some element of K which is distinct from p and whose upper distance from p is less than e.

In this section it will be shown that if, in a plane S, G is any upper semi-continuous collection of mutually exclusive bounded continua such that every point of S belongs to some continuum of the collection G and no continuum of G separates S, then if each continuum of G is considered as a point, and the term region is suitably defined, all the Axioms 1–8 of the author's article‡ *On the foundations of plane analysis situs* hold true, if the space S of that article is interpreted to mean the collection of elements G. Thus the set of elements G is, with respect to the notion of limit point defined in that paper, *topologically equivalent to the set of ordinary points in a plane* S. Furthermore the notion of limit point so defined coincides, for the case of an upper semi-continuous collection, with the natural interpretation of limit element given above.

From here on, in this section, it is understood that there has been selected some definite upper semi-continuous collection of bounded continua

* Presented to the Society, December 30, 1924.

† If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $[l(PN)]$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. It is to be observed that $u(MN)$ may be different from $u(NM)$. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N. According to this terminology N is not always at the upper distance $u(MN)$ from M.

‡ These Transactions, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as F. A.

such that no one of these continua separates S and such that every point of S belongs to just one of them, and the letter G will be used throughout to denote this particular upper semi-continuous collection.

DEFINITIONS. A set of elements of the set G is said to be the *sum* of the sets K_1 and K_2 if every element which belongs to K_1 or to K_2 belongs also to K and every element which belongs to K belongs either to K_1 or to K_2. Two sets of elements of G are said to be *mutually exclusive* if there is no element of G which belongs to both of them. Two sets of elements of G are said to be *mutually separated* if they are mutually exclusive and neither of them contains a limit element of the other one. A set of elements G is said to be connected if it is not the sum of two mutually separated sets of elements of G. A set of elements of G is said to be *closed* if it contains all its limit elements. A set of elements of G is said to be a *continuum* of elements of G if it is both closed and connected. A set K of elements of G is said to be *bounded* if the point set obtained by adding together the points of all the elements of G is a bounded set of points

DEFINITION. A bounded subset H of the set G is said to be a *simple closed curve* (of elements of G) provided it is disconnected by the omission of any two of its elements.

DEFINITION. If A and B are two distinct elements of G, and H is a closed, connected and bounded set of elements of G, and A and B belong to H, and H is disconnected by the omission of any one of its elements except A and B, then H is said to be a *simple continuous arc* (of elements of G) from A to B, and A and B are said to be the *extremities,* or the *end-elements,* of this arc.

DEFINITION. A *domain* of elements of G is a connected set D of elements of G such that for every element x belonging to D there exists a positive number d such that if y is any element of G at an upper distance from x less than d then y belongs to D.

DEFINITION. By the *boundary* of a set H of elements of G is meant the set of all elements $[x]$ of H such that x either belongs to H and is a limit element of the set $G—H$ or x does not belong to H but it is a limit element of H.

DEFINITION. A domain of elements of G is said to be a *complementary* domain of a closed set H of elements of G provided the boundary of D is a subset of H.

DEFINITION. The *outer* boundary of a bounded domain D of elements of G is the boundary of the unbounded complementary domain of the boundary of D.

NOTATION. In this paper if a letter is used to denote a set of elements belonging to G then the same letter with a bar above it will denote the set of *points* obtained by adding together the points of all the elements of that set of elements.

THEOREM 1. *If K is a set of points and H is the set of all elements $[g]$ of the collection G such that g contains at least one point of K, then H is closed if K is closed and H is connected if K is connected.*

Proof. I. Suppose that K is closed. Suppose that p is a limit element of H. Then for every positive integer n there exists an element p_n belonging to H and such that each point of p_n is at a distance less than $1/n$ from some point of p, and such that if $i \neq j$ then p_i is distinct from p_j. But each p_n contains a point P_n belonging to K. For each n there exists in p a point X_n such that the distance from P_n to X_n is less than $1/n$. Since p is closed and bounded the sequence $X_1, X_2, \cdots$ contains a subsequence of distinct points which has as its sequential limit point some point X in p. The point X is a limit point of $P_1, P_2, \cdots$. Since every P_n belongs to K and K is closed, therefore X belongs to K. Thus p contains a point of K and therefore p belongs to H. Thus H contains all of its limit elements. In other words it is closed.

II. Suppose that K is connected. Then H is connected. For suppose, on the contrary, that H is the sum of two mutually separated sets of elements H_1 and H_2. Let K_1 denote the set of points common to K and $\overline{H}_1$ and let K_2 denote the set common to K and $\overline{H}_2$. Since K is connected, either K_1 contains a limit point of K_2 or K_2 contains a limit point of K_1. Suppose that K_1 contains a point P which is a limit point of K_2. Let p denote that element of H_1 which contains p. Since G is an upper semi-continuous collection, if e is a positive number there exists a positive number d such that if an element of H contains one point whose distance from P is less than d then *every* point of that element is at a distance less than e from some point of p. But, since P is a limit point of K_2, K_2 contains a point P_d at a distance from P less than d. Hence if h denotes that element of H_2 which contains P_d then every point of h is at a distance less than e from some point of p. Thus p is a limit element of H_2, contrary to the supposition that H_1 and H_2 are mutually separated. A similar contradiction would be obtained if it were supposed that K_2 contains a limit point of K_1. Thus the supposition that H is not connected leads to a contradiction.

THEOREM 2. *If D is a bounded complementary domain of a bounded continuum of elements of G, and K is the outer boundary of D, and p is an element of K, then K is a continuum of elements of G and $K-p$ is connected.*

Proof. Let E denote the unbounded complementary domain of the boundary of D and let B denote the set of *points* which constitutes the boundary of $\overline{E}$. Since each point of B belongs to some element of K and each element of K contains a point of B and B is a closed and connected set of points, therefore, by Theorem 1, K is a closed and connected set of elements of G. I will proceed to show that if p is an element of K then $K-p$ is connected. Suppose, on the contrary, that $K-p$ is the sum of two mutually separated sets of elements H and N. Then clearly $H+p$ and $N+p$ are both closed and connected sets of elements and they have in common only the element p. Let x and y denote elements belonging to D and E respectively. Let X and Y denote points belonging to x and y respectively. Since the continua $\overline{N}+\overline{p}$ and $\overline{H}+\overline{p}$ have in common only the continuum $\overline{p}$, and $\overline{N}+\overline{p}+(\overline{H}+\overline{p})$ separates X from Y, therefore* either $\overline{N}+\overline{p}$ or $\overline{H}+\overline{p}$ separates X from Y. Suppose that $\overline{H}+\overline{p}$ does. Then $H+p$ separates x from y, that is to say $G-(H+p)$ is the sum of two mutually separated sets of elements of G such that one of these sets contains x and the other one contains y. Let D_x and D_y denote the complementary domains of $H+p$ that contain x and y respectively. Clearly D_x contains D. Let q denote an element of G that belongs to the set N. The element q is a limit element of D and therefore of D_x. But q does not belong to the boundary of D_x. Hence it belongs to D_x. But q is also a limit element of E. Thus D_x contains an element of E and therefore, since E is connected and contains no element of the boundary of D_x, E is a subset of D_x. Thus y belongs to D_x, contrary to supposition. Similarly the supposition that $\overline{N}+\overline{p}$ separates X from Y would lead to a contradiction. The truth of Theorem 2 is therefore established.

DEFINITION. A *region* (of elements of G) is a bounded domain (of elements of G) which has a connected boundary.

THEOREM 3. *If p is an element of G and e is a positive number there exists a region of elements of G such that every element of this region is at an upper distance less than e from the element p.*

Proof. Since the set of points p does not separate S there exists† a simple closed curve (of ordinary *points*) J enclosing p and such that every point on or within J is at a distance less than e from some point of p. Let H denote the set of all elements $[x]$ of G such that the point set x contains at least one point of J. The point set $\overline{H}$ is a continuum. Let D denote that complementary domain of $\overline{H}$ which contains the point set p and let B denote the boundary of D. Let R denote the set of all elements $[g]$ of G such that the point set g is a subset of D. By a theorem of Brouwer's,‡ B is a closed and connected set of points. But

* Cf. S. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczno-Fizyczne, vol. 26 (1913).

† See Theorem 1 of my paper *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ L. E. J. Brouwer, Mathematische Annalen, vol. 69.

the boundary of R consists of all those elements $[g]$ of G such that g contains a point of B. It follows that the boundary of R is connected. But clearly R is a domain. Hence it is a region. Every element which belongs to it is at an upper distance less than e from the element p.

THEOREM 4. *If p is an element of G and K is a set of elements of G then p is a limit element of K if and only if every region (of elements of G) which contains p contains at least one element of K which is distinct from p.*

The truth of Theorem 4 may be easily established with the help of Theorem 3.

By methods largely analogous to those used in a similar connection in my paper *Concerning the prime parts of a continuum** it may be shown that if the word "point" as used in F. A. is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements of G) and the word "region" as used therein is interpreted to mean "region of elements of G" as defined above, then, for the space $S(G)$ consisting of all such "points" (elements of G), Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true and consequently Theorems 1–15 of that paper all hold true. For an indication of material sufficient for a proof, based on this result, of the following Theorems 5, 6 and 7, see page 139 of F. A. and page 342 of my paper *Concerning simple continuous curves*.†

THEOREM 5. *No simple continuous arc of elements of G is disconnected by the omission of either of its extremities.*

THEOREM 6. *If K is a simple continuous arc of elements of G, then every closed and connected subset of K which contains more than one element is itself a simple continuous arc of elements of G.*

THEOREM 7. *If pq is a simple continuous arc of elements of G with p and q as its extremities, x is an element of G belonging to the set pq, but distinct from p and q, and H is a set of elements of G belonging to pq, then p is a limit element of the set H if and only if every simple continuous arc of elements of G which contains x and is a subset of pq (but does not have x as an extremity) contains at least one element of the set H which is distinct from x.*

THEOREM 8. *If K is a closed, connected and bounded set of elements of G, and H is a connected proper subset of K, then the set $K-H$ contains an element of G whose omission does not disconnect K.*

Proof. Suppose, on the contrary, that K is disconnected by the omission of any element of $K-H$. Let p denote some definite element of $K-H$. Then $K-p$ is the sum of two mutually separated sets of elements. Since H is connected and does not contain p it must belong

* Mathematische Zeitschrift, vol. 22 (1925), pp. 307—315.

† These Transactions, vol. 21 (1920), pp. 333—347.

wholly to one of these sets. Let K_1 denote the one of which H is a subset and let K_2 denote the other one. It is easy to see that K_1+p and K_2+p are both closed and connected. The set K_2 contains some element whose omission does not disconnect K_2+p. For otherwise, if q denotes some element of K_2, K_2+p would be a simple continuous arc having p and q as its extremities and therefore, by Theorem 5, K_2+p would not be disconnected by the omission of q, contrary to supposition. Let r denote an element of K_2 whose omission does not disconnect K_2+p. Since K_2+p-r and K_1+p are connected sets having the element p in common their sum is connected. But their sum is $K-r$. Thus K is not disconnected by the omission of the element r. But since r belongs to K_2 it belongs to $K-H$.

With the help of Theorems 2 and 6 the following theorem may be established by an argument largely similar to that used in the proof of Lemma 8 of my paper *Concerning the prime parts of certain continua which separate the plane.**

THEOREM 9. *If pq is a simple continuous arc of elements of G, then $G-pq$ is connected.*

THEOREM 10. *If J is a simple closed curve of elements of G and p and q are distinct elements of J, then J is the sum of two simple continuous arcs (of elements of G) which have p and q as their extremities and which have in common no element except p and q.*

In view of results established above Theorem 10 is a consequence of Theorem 4 of my paper *Concerning simple continuous curves.*†

THEOREM 11. *If J is a simple closed curve of elements of G, then $G-J$ is the sum of two domains (of elements of G). Only one of these domains is bounded and J is the boundary of each of them.*

Proof. Let p and q denote two distinct elements of G belonging to the set J. By Theorem 10, J is the sum of two simple continuous arcs a and b which have p and q as their extremities but which have in common no other element of G. By Theorem 9 neither a nor b separates G and therefore neither $\bar{a}$ nor $\bar{b}$ separates S. But the common part of $\bar{a}$ and $\bar{b}$ consists of two mutually exclusive continua $\bar{p}$ and $\bar{q}$. It follows, by a theorem of Miss Mullikin's,‡ that $S-J$ is the sum of two mutually exclusive domains. It is easy to see that one of these domains (call it D_1)

*Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 170–175. In this paper in the statement of Lemma 5 replace the last "M" by "K". Hereafter this paper will be referred to as P. C. S.

† Loc. cit.

‡ Cf. her thesis *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

is bounded and the other one (D_2) is unbounded. Let I denote the set of all those elements of G which are subsets of D_1 and let E denote the set of all those which are subsets of D_2. Clearly I and E are domains of elements of G and $I+E = G-J$. Let B denote the boundary of I. If B is not identical with J then it is a proper subset of J. It easily follows, with the help of Theorems 10 and 6, that B is a simple continuous arc of elements of G. But B separates G and, by Theorem 9, no arc separates G. Thus the supposition that J is not the boundary of I leads to a contradiction. In a similar way it may be proved that J is also the boundary of E.

Definition. Of the two domains which are complementary to a simple closed curve of elements of G, the bounded one will be called the *interior*, and the unbounded one will be called the *exterior*, of that curve.

Theorem 12. *If D_1 and D_2 are bounded domains of elements of G, and D_1 has a connected boundary, and the boundary of D_2 is a subset of D_1, then D_2 is a subset of D_1.*

Proof. Since D_2 is bounded it has at least one boundary element. But every one of its boundary elements belongs to D_1. Hence D_1 contains at least one element of D_2. Suppose D_2 is not a subset of D_1. Then it must contain an element which does not belong to D_1. It follows that D_2 contains an element of B_1, the boundary of D_1. But B_1 is connected and contains no point of the boundary of D_2. Hence B_1 is a subset of D_2. Thus D_2 contains the boundary of E, the unbounded complementary domain of B_1. Therefore D_2 contains an element of E. But E is connected and contains no point of the boundary of D_2. Hence E is a subset of D_2, contrary to the hypothesis that D_2 is bounded.

Theorem 13. *If R is a region (of elements of G) and k is either a single element of G or a simple continuous arc of elements of G every element of which (except possibly just one of its end elements) belongs to R, the set of all those elements of R which do not belong to k is a domain of elements of G.*

Proof. Let B denote the boundary of R. By Theorem 9, k does not separate G. Hence $\overline{k}$ does not separate S. Furthermore, if x and y are any two elements of R, $\overline{x}$ and $\overline{y}$ are not separated from each other by $\overline{B}$. Also, either $\overline{k}$ and $\overline{B}$ have no point in common or their common part is a bounded, closed and connected point set consisting of all the points of a certain single element of G. It follows* that $\overline{k}+\overline{B}$ does not separate $\overline{x}$ from $\overline{y}$. Hence $k+B$ does not separate x from y. Hence the set of all those elements of R which do not belong to k is connected. It easily follows that it is a domain.

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* See S. Janiszewski, loc. cit.

THEOREM 14. *If R is a region of elements of G there exists a simple closed curve of elements of G such that every element of G which belongs to this curve is an element of R.*

Proof. Let p and q denote two distinct elements of the region R. There exists* a simple continuous arc pq of elements of G such that every element of pq belongs to R. Let r denote some element of pq distinct from p and from q. By Theorem 13, $R-r$ is a domain. Hence there exists a simple continuous arc pyq which is a subset of $R-r$ and has p and q as its end elements. It is easy to see that the sum of the arcs pxq and pyq contains as a subset a simple closed curve of elements of G.

THEOREM 15. *If pxq and pyq are simple continuous arcs (of elements of G) which have p and q as their extremities but which have in common no other element of G, and J is the simple closed curve formed by these two arcs, and pzq is a simple continuous arc (of elements of G) every element of which, except p and q, belongs to R, the interior of J, and J_1 denotes the simple closed curve formed by the arcs pxq and pzq and J_2 denotes the one formed by pyq and pzq, then* (1) *R_1, the interior of J_1, is a subset of R,* (2) *pyq is, except for p and q, wholly in the exterior of R_1,* (3) *R_1 has no point in common with R_2, the interior of J_2.*

Theorem 15 may be proved by an argument closely analogous to that used to prove Theorem 24 of F. A. In the proof there given reference is made to Theorem 21 of F. A. For the case where K and R are interiors of simple closed curves of elements of G this Theorem 21 may be easily proved with the help of Theorem 11 above.

THEOREM 16. *Under the same hypothesis as in Theorem* 15, *R is the sum of R_1, R_2, and $pzq-(p+q)$.*

Theorem 16 may be proved by an argument closely parallel to that employed in F. A. to prove Theorem 25.

THEOREM 17. *If p and q are two distinct elements of G and pxq, pyq and pzq are simple continuous arcs of elements of G no two of which have in common any element except their extremities (p and q) and J_1, J_2 and J_3 are the simple closed curves formed by these arcs taken in pairs, then the interiors of J_1, J_2 and J_3 are not mutually exclusive.*

Theorem 17 may be proved with the help of Theorems 15 and 16 by a method similar to that used in F. A. to prove Theorem 26 with the aid of Theorems 24 and 25.

THEOREM 18. *If pxq and pyq are simple continuous arcs (of elements of G) which have in common only their extremities p and q, J is the simple closed curve formed by these arcs, and pzq is an arc which lies, except for*

* See Theorem 15 of F. A.

its extremities, entirely in the exterior of J, then (1) *either y is without J_1, the simple closed curve formed by pxq and pzq, or x is in the exterior of J_2, the simple closed curve formed by pyq and pzq,* (2) *if y is without J_1 then x is in the interior of J_2 and the interior of J_2 is the sum of the interior of J, the interior of J_1 and the set of elements $pxq-(p+q)$.*

Theorem 18 may be proved with the help of Theorems 16 and 17 by a method analogous to that used in F. A. to prove Theorem 27 with the aid of Theorems 25 and 26.

THEOREM 19. *If R is a region (of elements of G) and p is an element of R, then there exists a simple closed curve of elements of G which lies in R and whose interior contains p and is a subset of R.*

With the use of Theorems 11, 12, 13, 17 and 18, Theorem 19 may be proved by an argument closely analogous to that employed to prove Theorem 36 in F. A.

DEFINITION. A set R of elements of G will be said to be a *region in the restricted sense* if and only if it is the interior of some simple closed curve of elements of G.

THEOREM 20. *If p is an element of G and H is a set of elements of G then p is a limit element of H if and only if every region in the restricted sense that contains p contains also an element of H disitinct from p.*

Suppose first that p is a limit element of H and that R is a region in the restricted sense which contains p. Since R is also a region in the original sense, therefore, by Theorem 4, R contains an element of H distinct from p.

Suppose, secondly, that every region in the restricted sense which contains p contains an element of H distinct from p. If R is a region in the original sense that contains p, then, by Theorem 19, there exists a region in the restricted sense which contains p and which is a subset of R. Since every such region contains an element of H distinct from p, therefore so does R. Hence, by Theorem 4, p is a limit element of H.

THEOREM 21. *If the word "point", as used in F. A., is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements G) and the word "region", as used therein is interpreted to mean "region in the restricted sense", as defined above, then for the space $S(G)$ consisting of all such "points" (elements of G) Axioms 1–8 of F. A. all hold true.*

Proof. It has been established that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true for the set of elements G provided *region* is interpreted as defined near the beginning of the present section. With the help of Theorems 19 and 20 it is easy to see that these axioms continue to hold true if region is interpreted in the restricted sense as defined above. That

Axiom 3 holds true for this interpretation can easily be seen with the help of Theorem 11. The truth of Axioms 6′ and 7′ of the system Σ_2 (see page 163 of F. A.) is a consequence of Theorems 11 and 13. Thus all the axioms of the system Σ_2 hold true for G. With the aid of results established in my paper *Concerning a set of postulates for plane analysis situs** and the fact that every region in the restricted sense is the interior of a simple closed curve it easily follows that Axiom 6 and 7 also hold true.

THEOREM 22. *Between the continua of the upper semi-continuous collection G and the points of an ordinary euclidean plane S there is a one to one correspondence (with single valued inverse) which preserves limits, that is to say which has the property that a point P in S is a limit point of a point set M in S if and only if the element of G which corresponds to P is a limit element (in the sense of the definition given in this paper) of the set of elements corresponding to M.*

The truth of Theorem 22 follows from Theorems 20 and 21 and the results of my paper *Concerning a set of postulates for plane analysis situs.*

II. THE PRIME PARTS OF A BOUNDED CONTINUUM IN THE PLANE

Hans Hahn† has introduced the notion of *prime parts* of a continuum. If P is a point of a continuum M then by the prime part K_P (of M) is meant the set of all points $[X]$ belonging to M such that, for every positive number e, there exists a finite set of irregular points (of M), $X_1, X_2, X_3, \cdots, X_n$ such that

$$r(X, X_1) \leqq e, \quad r(X_1, X_2) \leqq e, \quad \cdots, \quad r(X_{n-1}, X_n) \leqq e, \quad r(X_n, P) \leqq e.‡$$

In my paper *Concerning the prime parts of a continuum*,§ I have shown that if a bounded continuum has more than one prime part then it is a

* These Transactions, vol. 20 (1919), pp. 169–178.

† *Über irreduzible Kontinua,* Sitzungsberichte der Königlichen Akademie der Wissenschaften zu Wien, vol. 130 (1921), pp. 217–250.

‡ A continuum M is said to be connected im kleinen (or regularly connected) at the point P if for every positive number e there exists a positive number d such that if X is a point of M at a distance from P less than d then X and P lie together in some connected subset of M of diameter less than e. Cf. Hans Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist,* Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 318–322; Pia Nalli, *Sopra una definizione di dominio piano limitato da una curva continua, senza punti multipli,* Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 391–401; S. Mazurkiewicz, *Sur les lignes de Jordan,* Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209. An *irregular* point of a continuum M is a point at which M is not regularly connected. If X and Y are two points, $r(X, Y)$ denotes the distance from X to Y.

§ Loc. cit. Hereafter this paper will be referred to as P. C.

continuous curve with respect to its prime parts considered as points. In this section I will establish the following more general theorem.

THEOREM 23. *If, in a plane S, M is a bounded continuum no prime part of which separates S, and every prime part of M is considered as an element, and every point which does not belong to M is considered as an element, then the collection G of all such elements is an upper semi-continuous collection, and between the elements of G and the points of a plane H there exists a one to one correspondence which preserves limits and which is such that the image in H of the set of all prime parts of M is a continuous curve.*

Proof. Suppose p is an element of G which is a prime part of M and suppose that e is a positive number. By Theorem 3 of P. C. there exists a positive number d, less than e, such that if q is any element of G which is a prime part of M and whose lower distance from p is less than d then its upper distance from p is less than e. If q is any element of G which is not a prime part of M then q is a point, and therefore its upper distance from p is the same as its lower distance and thus its upper distance from p is less than e if its lower distance from p is less than d.

Suppose that p is an element of G which is not a prime part of M. In this case p is a point and if e is any positive number and d is less than e and also less than the lower distance of p from M then every element of G whose lower distance from p is less than d is also a point and therefore its upper distance from p is less than d and therefore less than e.

It follows that G is an upper semi-continuous collection. The truth of the remainder of Theorem 23 easily follows with the help of Theorem 17 of P. C. and Theorem 22 of the present paper.

Thus, regarded as being composed of its prime parts as elements, *every bounded continuum* (no one of whose prime parts separates its plane) *is a continuous curve,* not only as far as its internal structure is concerned, but *also as far as its relation to the remainder of the plane is concerned.**

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* Professor R. L. Wilder has called my attention to the fact that the statement in line 10 of page 172 of P. C. S. to the effect that q contains no point of the boundary of D_x is incorrect and that Lemma 3 appears to be false. This is indeed the case if in the definition (given near the top of page 171) of the outer boundary of D *with respect to the prime parts of M* the phrase "boundary of D" is interpreted to mean the boundary, in the ordinary sense, of the *set of points D*. If, however, this phrase is interpreted to mean the boundary (as defined in Section I of the present paper) of the domain D, the points of $S-M$ (and therefore, in particular, those of D) and the prime parts of M being the elements of G, then Lemma 3 holds true and the argument given in P. C. S. to prove it holds good except that the first sentence of this argument is to be replaced by the following: "Let B denote the point set obtained by adding together all the prime parts of M which belong to K." The fourth paragraph of page 171 of P. C. S. is to be replaced by the following statement: "In view of Lemma 2 it is clear that every prime part of M which

III. The maximal connected subsets of a closed plane point set

A maximal connected subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

Theorem 24. *If, in a plane S, M is a closed point set and every maximal connected subset of M is considered as an element and every point which does not belong to M is considered as an element, then the set G of all such elements is an upper semi-continuous collection.*

Proof. Let G_1 denote the set of all maximal connected subsets of M and let G_2 denote the set of all points which do not belong to M.

Suppose p is an element of G_1. Let e denote a positive number. There exists a positive number d such that if q is an element of G at a lower distance from p less than d then q is at an upper distance from p less than e. For suppose this is not the case. Then there exists a positive number e and an infinite sequence of distinct elements (of the set G) p_1, p_2, $p_3, \cdots$ such that, for every n, p_n contains two points B_n and C_n such that B_n is at a distance less than $1/n$ from some point A_n which belongs to p, while C_n is at a distance greater than e from every point of p. There exist two points B and C and a sequence of distinct integers n_1, n_2, $n_3, \cdots$ such that B is the sequential limit point of the sequence B_{n_1}, B_{n_2}, $B_{n_3}, \cdots$ and C is the sequential limit point of the sequence C_{n_1}, C_{n_2}, $C_{n_3}, \cdots$. The limiting set of the sequence p_{n_1}, p_{n_2}, $p_{n_3}, \cdots$ is a closed and connected point set K which contains B and C. But clearly B belongs to p and C does not. Since the continua p and K have B in common, their sum is connected. But their sum is a subset of M and it contains a point C not belonging to p. Thus p is not a maximal connected subset of M. But this is contrary to hypothesis.

The case where p is an element of G_2 may be treated by a method analogous to that employed in a similar connection in the proof of Theorem 23 in Section II. The truth of Theorem 23 is therefore established.

As a consequence of Theorems 24 and 22 we have the following result.

Theorem 25. *If, in a plane S, M is a closed and bounded point set no subset of which separates S, and every maximal connected subset of M is considered as an element, and every point which does not belong to M is considered as an element, then the set of all such elements is topologically equivalent to the set of all points in a plane.*

IV. Upper semi-continuous collections in space of n dimensions

Theorem 26. *If, in a euclidean space S of any number of dimensions, G is an upper semi-continuous collection of mutually exclusive continua such*

belongs to the set K defined above contains at least one point of the point set which forms the outer boundary, in the ordinary sense, of the domain (of points) D."

that every point of S belongs to some continuum of the collection G, then (regardless of whether the continua of G separate S) the term region may be so defined that (1) *if the elements of G are called points then Axioms* 1, 2, 4 *and* 5 *and Theorem* 4 *of F. A. all hold true and furthermore an element p of G will be (in the sense defined in* I) *a limit element of a set H of elements of G if and only if every region that contains p contains at least one element of H distinct from p.*

I will proceed to indicate how this theorem may be established.

Let us retain all the definitions given, in Section I, before the statement of Theorem 1. But let the definition of a region of elements of G, as given in that section, be replaced by the following:

DEFINITION. If p is an element of G and e is a positive number, then by R_{pe} is meant the set of all elements q of G such that q and p belong to some connected set of elements of G such that every element of this connected set is at an upper distance from p less than e. For every p and e the set R_{pe} is called a *region* of elements of G.

It is easy to see that Theorems 1, 3 and 4 of Section I hold true here and that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. may be established as indicated after the statement of Theorem 4 in Section I.

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